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Swarm dynamics, attractors and bifurcations of active Brownian motion

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Abstract. We apply the model of active Brownian dynamics to describe the motion of swarms of animals. The interaction between the objects is decribed by simple models. First we apply the model of a rigid body which reduces the dynamics to a one-body problem. Next we treat the model of global coupling. It is shown that then the problem is reduced to a center of mass and relative motion problem, i.e. effectively to a two-body problem. The attractors corresponding to translational and rotational motions of the swarm are identified. The dynamic and stochastic bifurcations between the rotational and translational modes are analyzed.

1 Characteristics of the dynamics of swarms

This paper is devoted to the investigation the dynamics and stochastic theory of simple dynamical models of moving swarms of animals. We use here the general notation of "swarms" for confined systems of particles (or more general objects) collective motion far from equilibrium. The study of objects like swarms of animals is a traditional object of biological and ecological investigations [1] but still a rather young field of physical studies (see e.g. Refs. [2–5]).

Since the dynamics of swarms of driven particles has captured the interest of theorists, many interesting effects have been revealed and in part already explained. We mention the comprehensive survey of Okubo and Levin [1] on swarm dynamics in biophysical and ecological respect. Further we mention the survey of Helbing [2] covering traffic and related self-driven many-particle systems and the comprehensive books of Vicsek [3], Mikhailov and Calenbuhr [4], and of Schweitzer [5]. In the book of Okubo and Levin [1] we find a classification of the modes of collective motions of swarms of animals. It is discussed that animal groups have three typical modes of motion:

1. translational motions,
2. rotational excitations and
3. amoeba-like motions.

This classification has been verified for example in experiments by Ordemann, Balazsi and Moss [6]. In studies of the motion of daphnia these authors have shown that, depending on the existence of a external light source a whole swarm of these animals may switch from a translational type of motion to a very correlated type. In the correlated mode the whole swarm starts to rotate around the light shaft. The transition from translational modes to rotational modes has also been observed in the swarm motion of fishes and birds.

At present it seems to be impossible to describe the rich variety of possible collective motions observed in nature. Instead we will study in the following the collective modes and the

distribution functions of two simple models. We investigate finite systems of particles confined by attracting forces which are self-propelled by active friction and parallelized by small velocity-dependent interactions. In our first model we describe the swarm as a rigid body and in the second model we map it to a system with linear attractive interactions - springs. These two ways of description may be considered as rough models for the collective motion of nonequilibrium clusters and of swarms of cells and organisms as well [7–13]. For alternative models based on velocity-velocity interactions see Refs. [3,14,15]. From the point of view of statistical mechanics the main purpose of this work is the study of the dynamics of active Brownian particles including interactions. The self-propelling of the particles is modelled by active friction as introduced first by Rayleigh [16] and studied in the context of Brownian motion by Klimontovich [17]. The interaction between the particles is modelled by potential forces and a crude model of parallelizing interactions. The consideration is mostly restricted to $2-d$ models. Driving the system by negative friction we bring the system to far from equilibrium states. Earlier studies have shown that driven interacting systems may have many attractors and that noise may lead to transitions between the deterministic attractors [7,9].
We will show here that the collective motion of swarms (large clusters) of driven Brownian particles reminds very much the typical modes of parallel motions in swarms of living entities. In the largest part of the study we will use the method of global coupling introduced in this context in [7,9] which allows to reduce the problem effectively to a two-particle problem.

2 General model of the dynamics of swarms

We introduce conservative interactions described by the potential $U(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ and postulate a dynamics of Brownian particles determined by the Langevin equation:

$$\dot{\mathbf{r}}_i = \mathbf{v}_i \,; \qquad m\dot{\mathbf{v}}_i = \mathbf{F}_i - \nabla U(\mathbf{r}_1,\ldots,\mathbf{r}_N) + \sqrt{2D}\xi(t) \tag{1}$$

where $\xi(t)$ is a stochastic force with strength D and a δ-correlated time dependence.

$$\langle \xi_i(t)\rangle = 0\,; \qquad \langle \xi_i(t)\xi_j(t')\rangle = \delta(t-t')\delta_{ij} \tag{2}$$

The non-dissipative forces are assumed to be reperesented by external and interaction forces which are pairwise additive

$$U = \sum_i U(\mathbf{r_i} + \sum_{ij} \varPhi(r_{ij}) \tag{3}$$

In most part of this work we will use only parabolic approximations of the interaction forces. The dissipative forces are expressed in the form

$$\mathbf{F}_i = -m\gamma(\mathbf{v}_i^2)\mathbf{v}_i - m\sigma(\mathbf{v}_1,..,\mathbf{v_N}) \tag{4}$$

The coefficient γ denotes a velocity-dependent friction, which possibly has a negative part. This type of driving forces was first use by Lord Rayleigh and later also in the context of the dynamics of cells and swarms [16,18–22]. In the case of thermal equilibrium systems we have $\gamma(\mathbf{v}) = \gamma_0 =$ const.. In the general case where the friction is velocity dependent we will assume that the friction is monotonically increasing with the velocity and converges to γ_0 at large velocities. In the following we will use the following ansatz based on the depot model for the energy supply [20–22]

$$\gamma(\mathbf{v}^2) = \left(\gamma_0 - \frac{dq}{c+dv^2}\right) \tag{5}$$

where c, d, q are certain positive constants characterizing the energy flows from the depot to the particle. Dependent on the parameters γ_0, c, d, and q the dissipative force function may have one zero at $\mathbf{v} = 0$ or two more zeros with

$$\mathbf{v}_0^2 = \frac{d}{c}\zeta\,; \qquad \zeta = \frac{qd}{c\gamma_0} - 1. \tag{6}$$

Here ζ is a bifurcation parameter. In the case $\zeta > 0$ a finite characteristic velocity v_0 exists. Then we speak about active particles. For $|\mathbf{v}| < v_0$, the dissipative force is positive, i.e. the particle is provided with additional free energy. Hence, slow particles are accelerated, while the motion of fast particles is damped (see Fig. 1). The asymptotics for large velocities is passive.

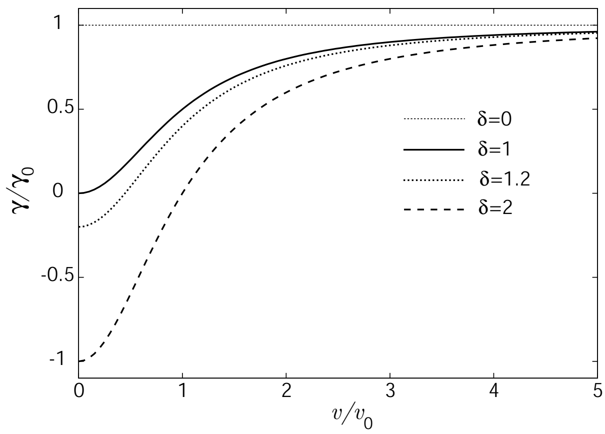

Fig. 1. The typical form of a friction function with active (negative) part at small velocities (parameter $\delta = \zeta + 1$).

A simpler expression for the the friction is obtained by a Taylor expansion cutted after the second term which leads to Rayleigh formula containing only 2 constants

$$\gamma(\mathbf{v}^2) = \left(-\alpha + \beta v^2\right) \tag{7}$$

The stationary velocity is

$$v_0^2 = \frac{\alpha}{\beta} \tag{8}$$

The second contribution to the dissipative forces expresses a velocity-dependent interactions in particular some tendency to synchronize the velocities. For simplicity we will assume the simple law

$$\sigma_i(\mathbf{v_1}, ..., \mathbf{v_N}) = -\frac{\sigma^2}{N}\sum_{ij}(\mathbf{v}_i - \mathbf{v}_j) = -\sigma^2(\mathbf{v}_i - \mathbf{V(t)}) \tag{9}$$

This models a dissipative force which tends to parallelize the individual velocities with the swarm velocity $\mathbf{V}(t)$ which is the velocity of the center of mass of the swarm.

$$\mathbf{V(t)} = \frac{1}{N}\sum_j \mathbf{v_j} \tag{10}$$

The present model of velocity-velocity interactions corresponds to a global velocity-coupling with infinite range. Corresponding couplings with finite range were studied Vicsek and collaborators [3]. There are other more complicated approximations to model the dissipative interactions. We mention Oseen-type interactions [11], several models studied by Vicsek and collaborators [3,14,15] and a model proposed by Mach and Schweitzer [25]. Here we restrict our investigation to the simplest model of velocity coupling - global velocity coupling. In our model the dynamics of the Brownian particles is determined by Langevin equations with dissipative contributions.

3 The model of rigid-oriented body dynamics

The most drastic simplicfication of the N-particle problem is the model of a rigid body with fixed orientation. This corresponds in particular to the limit $\sigma \to \infty$, which means that the particles cannot move relative to each other and are also not allowed to rotate around the center of mass. That the rigid body motion with fixed orientation is indeed one of the dynamic modes of N-body active swarms has been shown in several theoretical [23] and experimental [24] papers dealing with swarms of charged grains. In the work [23] many initial configurations of $N = 2, 3, 4, 5$ were simulated and it was demonstrated that a considerable part of the initial conditions lead automatically to a rigid body dynamics. In other words, the rigid-oriented type of swarm motion should be one of the attractors of the dynamics.
In the following we will discuss the dynamics of rigid swarms of active particles in a two-dimensional space, We consider in a first approximation a swarm without any particle interaction and without driving and noise. The center of mass is given by

$$\mathbf{R} = [X_1, X_2], \qquad X_1 = \frac{1}{N}\sum x_{i1}, \ X_2 = \frac{1}{N}\sum x_{i2}. \tag{11}$$

The free motion of the center of mass corresponds to

$$\mathbf{V} = \mathbf{V}(t=0) = const; \qquad \mathbf{R}(t) = \mathbf{V}(0)t + \mathbf{R}(0). \tag{12}$$

The relative motion of the deviations

$$\delta\mathbf{r}_i = \mathbf{r}_i - \mathbf{R}; \qquad \delta\mathbf{v}_i = \mathbf{v}_i - \mathbf{V} \tag{13}$$

satisfies the equations

$$\dot{\delta}\mathbf{r}_i = \delta\mathbf{v}_i; \qquad \dot{\delta}\mathbf{v}_i = -\sigma\delta\mathbf{v}_i \tag{14}$$

The solution is elementary and reads

$$\delta\mathbf{v}_i(t) = \delta\mathbf{v}_i(0)\exp(-\sigma t), \qquad \delta\mathbf{r}_i(t) = \delta\mathbf{r}_i(0) + \frac{\delta\mathbf{v}_i(0)}{\sigma}(1 - \exp(-\sigma t)). \tag{15}$$

In other words, after certain time all particles move parallel with the same velocity. The relative positions are also fixed, rotations with respect to the center of mass are also not admitted. We may speak about a rigid oriented body. In the general case with driving and including interaction forces, the swarm cannot be considered as a rigid oriented body, however we may still use this assumption as a rough model which is more or less true in the limit of very large parallelizing forces. We assume within this model that the relative positions within the swarm are fixed and that rotations are frozen in. That means we have a rigid body with fixed orientation in the space. The relative positions are determined by the interaction forces, for simplicity we may assume that the particles observe a distribution of relative equlibrium, i.e. they are in general Boltzmann-distributed

$$\rho(r_1, ...r_N) = const \exp[-\beta U(r_1, ..., r_N)]. \tag{16}$$

where U is the potential of the (conservative) interactions as defined by eq.(3). This is the simplest mode of a many-body sytem of active particles since in this mode the whole problem reduces to the dynamics of the center of mass, which is a one-particle problem including external fields. Let us consider a swarm of active Brownian particles which are by pairwise forces bound to a cluster which is rigid with respect to relative position and orientation. We consider now the free motion of the cluster by studying the dynamics of the center of mass in an external field. The relative motion under the influence of the interaction is completely neglected within this model. The free motion of the center of mass M is in 2D-space described by the model equations

$$\begin{aligned} \dot{X}_1 &= V_1 \qquad & M\dot{V}_1 &= -M\gamma\left(V_1, V_2\right)V_1 + \sqrt{2D}\xi_1(t) \\ \dot{X}_2 &= V_2 \qquad & M\dot{V}_2 &= -M\gamma\left(V_1, V_2\right)V_2 + \sqrt{2D}\xi_2(t) \end{aligned} \tag{17}$$

The stationary solutions of the corresponding Fokker-Planck equation for free motion reads for the model defined by eq.(1.5) [22]

$$P_0(\mathbf{V}) = C\left(1 + dV^2\right)^{(q/2D_v)} \exp\left[-\frac{\gamma_0}{2D_v}V^2\right] \tag{18}$$

where $D_v = D/m^2$. We study now the influence of external forces. The case of constant external forces was already treated by Schienbein et al. [18,19]. Symmetric parabolic external forces were studied in Refs. [21,22] and the non-symmetric case is being investigated in Ref. [26]. More complicated external fields were studied in detail in several papers [11,27–30].
For simplification we consider here only the case of parabolic external potential

$$U(X_1, X_2) = \frac{1}{2}m\omega_0\left(X_1^2 + X_2^2\right). \tag{19}$$

More general external forces were studied in [**?**]. First, we discuss the deterministic motion, which is described by four coupled first-order differential equations. The motion of the center of mass corresponds to the motion of 1 particle in a harmonic field:

$$\begin{aligned} \dot{X}_1 = V_1 \qquad \dot{V}_1 = -\gamma\,(V_1, V_2)\,V_1 - \omega_0^2 X_1 \\ \dot{x}_2 = v_2 \qquad \dot{V}_2 = -\gamma\,(V_1, V_2)\,V_2 - \omega_0^2 X_2 \end{aligned} \tag{20}$$

For this case we have shown earlier [21,22] that a limit cycle in the $4-d$ space is developed, which corresponds to left/right rotations with the frequency ω_0. The projection of this periodic motion to the planes $\{x_1, x_2\}$ and $\{v_1, v_2\}$ are circles

$$X_1^2 + X_2^2 = r_0^2; \qquad V_1^2 + V_2^2 = v_0^2. \tag{21}$$

The trajectories converge to limit cycles and the energy to

$$H \longrightarrow E_0 = mv_0^2 \tag{22}$$

This corresponds to an equal distribution between kinetic and potential energy. In explicite form we may represent the motion on the first (forward) limit cycle in the $4-d$ space by the 4 equations [22]

$$X_1 = r_0\,\cos(\omega_0 t + \Phi)\,;\; V_1 = -r_0\,\omega_0\,\sin(\omega_0 t + \Phi) \tag{23}$$

$$X_2 = r_0\,\sin(\omega_0 t + \Phi)\,;\; V_2 = r_0\,\omega_0\,\cos(\omega t + \Phi) \tag{24}$$

The frequency is given by the time the particle need for one period moving on the circle with radius r_0 with constant speed v_0. This leads to $\omega_0 = r_0/v_0$ and means that the particles oscillate with the frequency given by the linear oscillator frequency. The trajectory on the limit cycle defined by the above 4 equations is like a hula hoop in the $4-d$ space. The projections to the $x_1 - x_2$ space as well as the projections to the $v_1 - v_2$ space are circles. The projections to the subspaces x_1-v_2 and x_2-v_1 are like a rod. In the $4-d$ space the attractor has therefore the form of a hula hoop. A second limit cycle is obtained by reversal of the velocity. This second (backward) limit cycle forms also a hula hoop which is different from the first one, however both l.c. have the same projections to the $\{x_1, x_2\}$ and to the $\{v_1, v_2\}$ plane. The motion in the $\{x_1, x_2\}$ plane has the opposite sense of rotation in comparision with the first limit cycle. Therefore both limit cycles correspond to opposite angular momenta. $L_3 = +Mr_0v_0$ and $L_3 = -Mr_0v_0$. Applying similar arguments to the stochastic problem we find that the two hoop-rings are converted into a distribution looking like two embracing hoops with finite size, which for strong noise converts into two embracing tyres in the $4-d$ space. In order to get the explicite form of the distribution we may introduce amplitude–phase representations [22]. The probability crater is located above the two deterministic limit cycles on the sphere $r_0 = v_0/\omega_0$. Strictly speaking not the whole spherical set is filled with probability but only two circle-shaped subsets on it, which correspond to a narrow region around the limit sets. The full stationary probability has the form of two hula hoop distributions in the $4-d$ space. This was confirmed by simulations [22].

The projections of the distribution to the $\{x_1, x_2\}$ plane and to the $\{v_1, v_2\}$ plane are smoothed $2-d$-rings. The distributions intersect perpendicular the $\{x_1, v_2\}$ plane and the $\{x_2, v_1\}$ plane. Due to the noise the Brownian particles may switch between the two limit cycles, this means inversion of the angular momentum (direction of rotation) [22,26]. As a result rotating clusters are getting unstable similar to asymmetric driven oscillators [26]. We note that without external fields the rigid body model with fixed orientations leads just to a constant drift of the swarm $\mathbf{V} = const$. In the next section we consider the opposite limit: We neglect external forces and concentrate on the dynamics of the relative motion and the interplay between relative motion and translation; the velocity-coupling is assumed to be weak.

4 The model of harmonic swarms with global coupling

The concept of global coupling has proven to be very useful for the investigation of stochastic systems. I our case we will show that the apporach of global coupling in coordinate space allows to reduce the problem essentially to the analysis of the dynamics and stochastics of pairs of active Brownian particles. In the following we will reduce all interactions to linear attracting spring forces. [7–9]. We consider two-dimensional systems of N point masses m with the numbers $1, 2, \dots, i, \dots, N$. We assume that the masses m are connected by linear pair forces $m\omega_0^2 (\mathbf{r}_i - \mathbf{r}_j)$. The dynamics of the system is given by the following equations of motion

$$\frac{d}{dt}\mathbf{r}_i = \mathbf{v}_i\,;\; m\frac{d}{dt}\mathbf{v}_i + m\omega_0^2 (\mathbf{r_i} - \mathbf{R(t)}) = \mathbf{F}_i(\mathbf{v_i}) - \sigma^{\mathbf{2}}(\mathbf{v_i} - \mathbf{V(t)}) + \sqrt{\mathbf{2D}}\xi_{\mathbf{i}}(\mathbf{t}) \tag{25}$$

We have in this model a global coupling in the coordinate space proportional to ω_0^2 which tends to concentrate the swarm around the center of mass. Further we have in addition with the term proportional to σ^2 an -in general rather small- global coupling force tending to synchronize (parallelize) the velocities of the particles in the swarm. We start with an investigation of the translational mode of this system. For the mean velocity we find by summation and expanding around $\mathbf{V}$ in a symbolic representation ($m = 1$)

$$\frac{d}{dt}\mathbf{V} = \mathbf{F}(\mathbf{V}) + \frac{1}{2}(\delta\mathbf{v}) * \mathbf{F}^{''}(\mathbf{V}) * (\delta\mathbf{v}) + \dots \tag{26}$$

In the translational mode of this system all the particles form a noisy flock which moves with nearly constant velocity modulus

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{R}(0) + v_0\mathbf{n}, \qquad \mathbf{r}_i(t) - \mathbf{R}(t) = 0; \qquad i = 1, \dots, N \tag{27}$$

The direction $\mathbf{n}$ may change from time to time due to stochastic influences. For the case of Rayleigh-driving, the dynamical equations have a simpler form

$$m\frac{d}{dt}\mathbf{V} = \left[\alpha - \beta V^2\right]\mathbf{V} - \frac{\beta}{N-1}\sum_i (\delta v_i)^2\mathbf{V} - 2\frac{\beta}{N-1}\sum_i (\mathbf{V}\cdot\delta\mathbf{v}_i)\,\delta\mathbf{v}_i + \sqrt{2D}\xi_V(t) \tag{28}$$

$$m\frac{d}{dt}\delta\mathbf{v_i} + m\omega_0^2\delta\mathbf{r_i} = \left[\alpha - \beta\delta v_i^2 - \beta V^2\right]\delta\mathbf{v_i} - 2\beta(\mathbf{V}\cdot\delta\mathbf{v_i})\mathbf{V} - \sigma^2\delta\mathbf{v_i} + \sqrt{2D}\delta\xi_i(t) \tag{29}$$

where

$$\xi_V(t) = \frac{1}{N}\sum_i \xi_i(t), \qquad < \xi_{Vx}\xi_{Vy} >= \delta_{xy}\delta(t - t`) \tag{30}$$

$$\delta\xi_i = \xi_i - \xi_V, \qquad < \delta\xi_{ix}\delta\xi_{iy} >= \delta_{xy}(1 - \frac{1}{N})\delta(t - t`), \tag{31}$$

$$\langle\delta\xi_{ix}\delta\xi_{iy}\rangle = \delta_{xy}(\delta_{ij} - \frac{1}{N})\delta(t - t`) \tag{32}$$

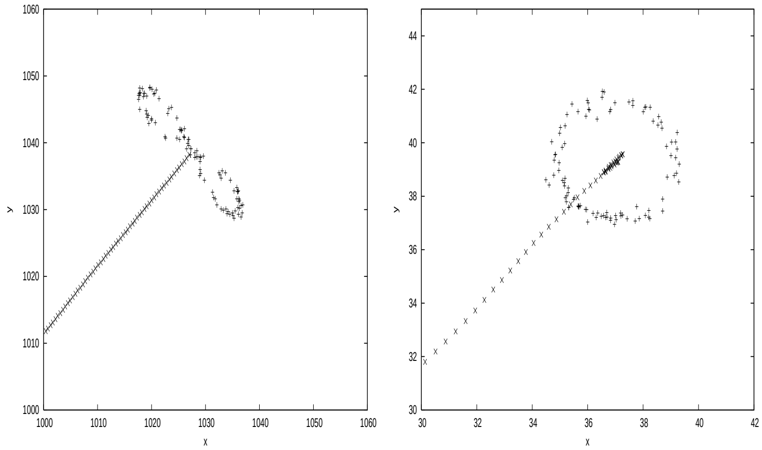

Fig. 2. The two basic configurations of a noisy system obtained by simulations of of $N = 100$ globally coupled active particles ($\omega_0^2 = 0.2, \sigma = 0$). Left panel: $D = 0.001$, translational mode , right panel: $D = 0.2$, rotational mode.

In order to find explicite results we simplify the equation for the mean momentum $\mathbf{V}$ assuming $m = 1$ and decoupling the dynamics of translational and relative motion

$$\frac{d}{dt}\mathbf{V} = (\alpha - \beta\mathbf{V}^2)\mathbf{V} - \frac{\beta}{N-1} < \sum_i (\delta v_i)^2 > \mathbf{V} - 2\frac{\beta}{N-1} < \sum_i (\mathbf{V}\delta\mathbf{v}_i)\,\delta\mathbf{v}_i > + \sqrt{2D}\xi_V(t) \quad (33)$$

In this way we decouple the center of mass motion from the relative motion. By averaging with respect to δv_i and neglecting the tensor character of the coupling to the relative motion we get

$$\frac{d}{dt}\mathbf{V} = (\alpha_1 - \beta\mathbf{V}^2)\mathbf{V} + \sqrt{2D}\xi_V(t) \quad (34)$$

Here $\alpha_1 < \alpha$ is determined by the mean quadratic dispersions. The distribution function of the flock is Boltzmann-like. The corresponding velocity distribution is

$$f^{(0)}(\mathbf{V}) = C \exp\left[\frac{1}{2D}\left(\alpha_1\mathbf{V}^2 - \beta\mathbf{V}^4\right)\right] \quad (35)$$

This way we find the most probable velocity

$$\mathbf{V_1}^2 = \frac{\alpha_1}{\beta} \quad (36)$$

The most probable velocity of the swarm is shifted to values smaller than for the free motion. The shift with respect to the free mode $V_0 = v_0\mathbf{n}$ depends on the noise strength D. The solution breaks down if the dispersion of the relative velocities δv^2 is so large that the linearization around V is no more possible. With increasing noise we find a bifurcation. This corresponds to the findings of Erdmann et al. [12]. The dispersion in the direction of the flight $\mathbf{V}$ is smaller than perpendicular to it.

5 Investigation of the dynamics of the relative motion of active particles

The relative motion with respect to the center of mass is completely symmetric with respect to the numbering of the particles i. In other words, there are no cross terms including two different particles as i, j. This is a unique property of global coupling, the relative dynamics

reduces completey to the binary problem.
Therefore we may concentrate our investigation on the study of 2 driven Brownian with a linear attracting force [26]. In this case we get by writing simply r, v instead of r_i, v_i

$$\frac{d}{dt}\mathbf{V} = \left[\alpha - \beta V^2\right]\mathbf{V} - \beta(\delta\mathbf{v})^2\mathbf{V} - 2\beta\left(\mathbf{V}\cdot\delta\mathbf{v}\right)\delta\mathbf{v} + \sqrt{2D}\xi_V(t) \tag{37}$$

$$\frac{d}{dt}\delta\mathbf{v} + \omega_0^2\delta\mathbf{r} = \left[\alpha - \beta\delta v^2 - \beta V^2\right]\delta\mathbf{v} - 2\beta(\mathbf{V}\cdot\delta\mathbf{v})\mathbf{V} - \sigma^2\delta\mathbf{v} + \sqrt{2D}\delta\xi(t) \tag{38}$$

We introduce an appropriate system of coordinates:
The center of mass $\mathbf{R}$ moves with the mass velocity $\mathbf{V}$. The relative motion under the influence of the forces is described by the relative radius vector $\delta\mathbf{r}$ and the relative velocity $\delta\mathbf{v}$. Since the vector V plays aspecial role we orientate the coordinate system relative to it $\delta\mathbf{r} = (x_1, x_2, x_3)$, $\delta\mathbf{v} = (v_1, v_2, v_3)$
where x_1, v_1 are the components in the direction of $\mathbf{V}$ and the two others perpendicular to it. This was we find the equations

$$\frac{d}{dt}\frac{V^2}{2} = V^2(\alpha - \beta V^2 - \beta\delta v^2 - 2\beta v_1^2), \tag{39}$$

$$\frac{d}{dt}\left[\frac{\delta v^2}{2} + \omega_0^2\frac{\delta r^2}{2}\right] = \delta v^2\left(\alpha - \sigma^2 - \beta V^2 - \beta\delta v^2 - 2\beta v_1^2\right), \tag{40}$$

$$\frac{d}{dt}\left[\frac{v_1^2}{2} + \omega_0^2\frac{x_1^2}{2}\right] = v_1^2\left(\alpha - \sigma^2 - \beta V^2 - \beta\delta v^2 - 2\beta V^2\right), \tag{41}$$

$$\frac{d}{dt}\delta\mathbf{r} = \delta\mathbf{v} \tag{42}$$

Writing the dynamical equations in this form of energy balances we see clearly that the most important physical processes are connected with the exchange of energy. There are two sets of basic attractors: The translational attractor with $V^2 = \alpha/\beta = v_0^2$. This attractor of motion corresponds to

$$\mathbf{V} = v_0\mathbf{n}; \qquad \mathbf{R}(t) = v_0\mathbf{n}t + \mathbf{R}(0); \qquad \delta\mathbf{v} = 0. \tag{43}$$

The other set of attractors corresponds to swarms at rest $\mathbf{V} = 0, |\delta\mathbf{v}| \simeq v_0$. The principal schema of the attractors is represented in Fig. 3. In order to study the dynamical problem in more detail we consider now the twodimensional case and $D = \sigma = 0$. Further we introduce the coordinates $z = V^2, x_1, x_2, v_1, v_2$ corresponding to the velocity of the center of mass squared, and the relative coordinates and relative velocities parallel and perpendicular to the velocity of the center of mass. Then we get the following 5 differential equations

$$\dot{z} = 2z(\alpha - \beta z - 3\beta v_1^2 - \beta v_2^2), \tag{44}$$

$$\dot{v}_1 = v_1(\alpha - \sigma^2 - 3\beta z - \beta v_1^2 - \beta v_2^2) - \omega_0^2 x_1, \tag{45}$$

$$\dot{v}_2 = v_2(\alpha - \sigma^2 - \beta z - \beta v_1^2 - \beta v_2^2) - \omega_0^2 x_2, \tag{46}$$

$$\dot{x}_1 = v_1, \tag{47}$$

$$\dot{x}_2 = v_2. \tag{48}$$

The qualitative analysis of this system of nonlinear o.d.e.'s shows at first that the system possesses a stable point attractor at $z = \alpha/\beta, v_1 = v_2 = x_1 = x_2 = 0$. The linear stability analysis provides the eigenvalues $-2\beta, -2\beta, -\sigma + i\omega_0, -\sigma - i\omega_0$. Therefore the point attractor is linearily stable provided $\sigma > 0$. The linear stability in the direction v_2 corresponding to the motion perpendicular to the translation is given only for $\sigma > 0$. However even at $\sigma = 0$ we still observe quadratic stability in this particular direction due to the terms $-\beta v_2^2$. These results illuminate the role of the velocity couplings. Without the existence of a (positive) velocity coupling, the swarms tends to show a weak instability in the transversal direction, i.e. it tends to

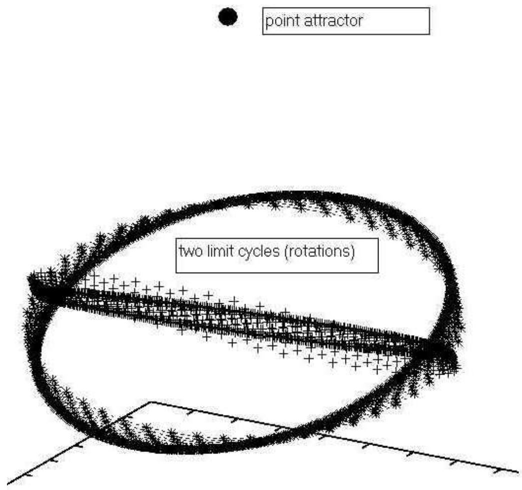

Fig. 3. Schema of the attractor structure in the phase space: There exist one stable point attractor $V = v_0$ (the fat point above) corresponding to the translational motion of the (tightly clustered and not rotating) swarm , and two limit cycles in the plane $V = 0$ (the two hoops below) corresponding to left/right rotations of the swarm at rest.

get broader and broader, remaining concentrated around the center of mass in the longitudinal direction.
An impression about the quite complicated attractor structure at $\sigma = 0$ we may get from representations of the vector fields of the velocities perpendicular to the motion of the center of mass. The vectorfields representing planes v_1, v_2 for different fixed values of $z = V^2$ which provide cross sections of the velocity fields for $\sigma = 0$ are shown in Fig. 4 More details about the dynamics we may obtain from explicite solutions, for the special set of parameters $\alpha = \beta = 1$. In Fig. 5 we show several solutions corresponding to initial conditions in the attractor region We see that typically (for different initial conditions within the attractor region) the relative velocity perpendicular to the swarm translation $v_2(t)$ decays very slowly, and the relative velocity $v_1(t)$ goes to zero in a very fast way. The velocity of the c.o.m. approaches α/β which here is "1" in a rather fast way, however a slow oscillatory contribution remains. Including a small amount of velocity synchronization $\sigma > 0$ all oscillatory components in the translational mode are damped out in the time $1/\sigma$. In the limit case $\sigma = 0$ i.e. no synchronization of the veloscities exist, the attracting region belonging to the stable point $z = 1$ is more complicated, because of the neutral stability of the v_2-dynamics. In spite of the fact that the attracting region is still located around the region $|z-1| << 1$, there are some more difficult features. For example, along the saddle-like curve $z = 1 - v_2^2, v_1 = x_1 = x_2 = 0$ possesses a long tongue reaching up to the neighborhood of the limit cycle attractors in the plane $z = 0$. In the case that synchronizing forces exist $\sigma > 0$, the curve $z = 1 - v_2^2$ looses its saddle character and the stable point at $z = 1$ is a standard node. Beside the attractor of translational motion at $z = 1$ we find other attractors in the plane $z = 0$ corresponding to rest of the center of mass. Indeed there are two attractors representing left/right rotations. This is shown in Fig. 6 In the attractor region of the rotational motion the translation decays completely to $V = 0$ and the sustained oscillations in the plane $V = 0$ form limit cycles. The two limit cycles in question have both the projections

$$x_1^2 + x_2^2 = 1; \qquad v_1^2 + v_2^2 = 1; \qquad z = 0 \tag{49}$$

Why the attractor is getting unstable if we introduce a translational motion. In order to understand this we introduce a small but finite translation $z = z_0 << 1$. Due to the structure of eqs.

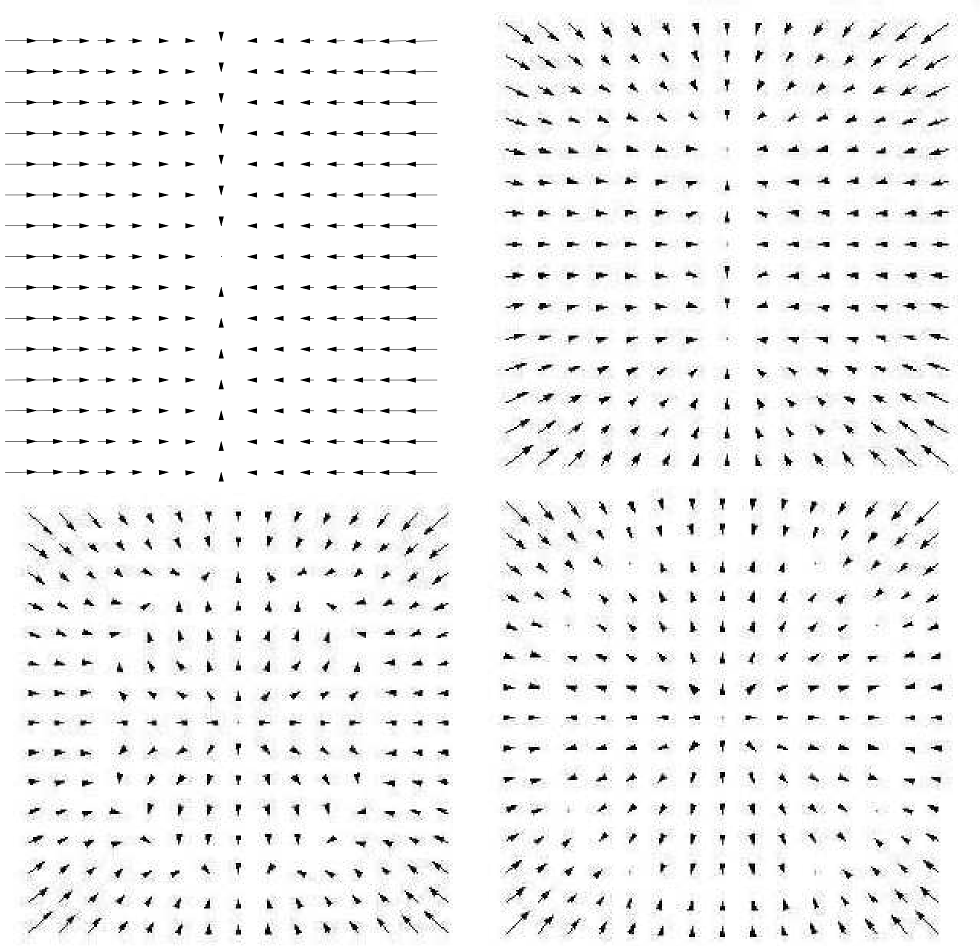

Fig. 4. Sequence of vectorfields of the relative velocities $v_2 - v_1$ for 4 cross-sections corresponding to decreasing values of the velocity of the mass center $V^2 \simeq 1, 0.8, 0.2, 0$ (top left to right down). We observe that the stable point corresponding to translations which is seen as an attracting point in the plane $z = 1$ converts in planes with smaller values of the translational velocity first for $z \simeq 0.8, 0.2$ to 2 saddle-like cross-sections and finally for the plane $z = 0$ to cross sections demonstrating the existence of a limit cycle.

(45-46) this leads immediately to a destruction of the rotational symmetry of the limit cycles, to an elliptic deformation with the longer axis in the direction perpendicular to the translation. As shown by Erdmann et al. [26,31] the loss of rotational symmetry leads to leaving an Arnold tongue of stability and consequently to a destruction of the limit cycles. We see that the rotations are indeed stable only in and near to the plane $z = 0$ i.e. for swarms at rest or near to the resting state.

Let us discuss again the important question about the influence of a finite but small effect of velocity synchronization ($\sigma > 0$). Studying the attractor structure we see easily that the curve $z = 1 - v_2^2, v_1 = x_1 = x_2 = 0$ is then no more corresponding to stationary states and loses its saddle character. For $\sigma > 0$ the point $z = \alpha/\beta$ gets linear stability. This way the strength of the attracting region of the corresponding point attractor $V^2 = v_0^2$ is increasing.

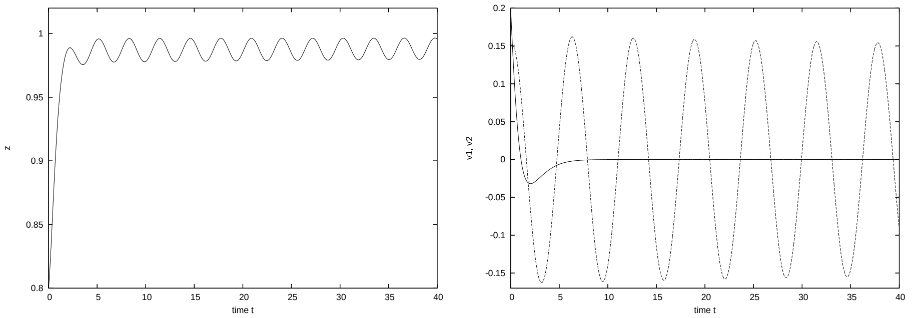

Fig. 5. Solutions of the o.d.e. for $\sigma = 0$ and initial conditions corresponding to the region of the attractor of translational motion, initial condition $z(0) = V^2(0) = 0.8$. Left panel: The translational velocity $z(t)$ approaches slowly the maximal value $z = 1$. An oscillating contribution remains which is due to the neutral stability of the $v_2(t)$ - dynamics. Right panel: The longitudinal component of the velocity $v_1(t)$ goes quickly to zero and the transversal velocity $v_2(t)$ decays very slowly or remains constant (because of neutral stability).

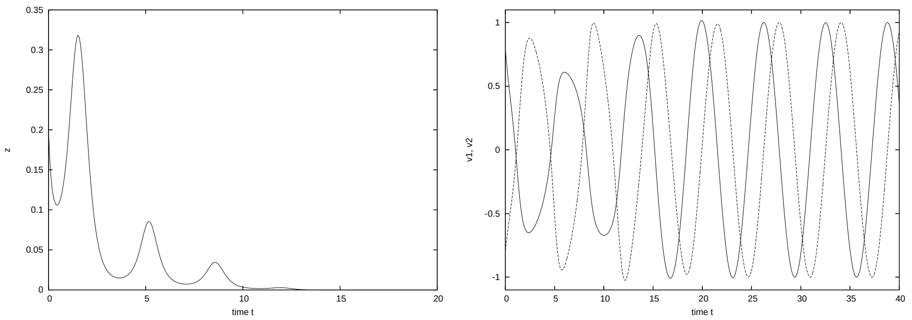

Fig. 6. Dynamics corresponding to the attractor region of the rotational motion of the pair: The left panel shows the decay of the velocity of translation $z(t) = V^2(t)$ to zero (rest state), and the right panel the formation of sustained oscillations of the longitudinal and transversal components of the relative velocities $v_1(t), v_2(t)$ (motion on a limit cycle).

6 Studies on the influence of noise

Including noise we expect instead of the point attractor at $V = v_0$ and the two line attractors - the 2 limit cycles in the plane $V = 0$ - that the dynamic systems forms some distributions around the attractors. At present we can only estimate the distributions. For this estimate we will use the approximation of independent dynamics of the velocity components v_1, v_2 and V. By the procedure of decoupling the stochastic equations we get for the mean velocity

$$\dot{V} = V(\alpha - \beta V^2 - 3\beta < v_1^2 > -\beta < v_2^2 >) + \sqrt{2D}\xi_V(t) \tag{50}$$

This leads to the distribution shown already in eq. (35) for the center of mass velocity. According to this distribution the most probable velocity is

$$V_1^2 = \frac{\alpha_1}{\beta}; \qquad \alpha_1 = \alpha - 3\beta < v_1^2 > -\beta < v_2^2 > \tag{51}$$

The distribution eq.(35) as well as the most probable velocity contain still an unknown constant α_1 which is determined by the distributions of the longitudinal and translational velocities. For the longitudinal fluctuations around the center of mass of the swarm near to the stable attractor $V^2 = \alpha/\beta$ we get

$$\frac{d}{dt}v_1 + \omega_0^2 x_1 = -2\alpha_1 v_1 + \sqrt{2D_r}\xi_1(t) \tag{52}$$

where $D_r = 2D$ is the noise strength for the relarive motion. This follows from the correlators given in eq. (31) for $N = 2$. The corresponding Fokker-Planck equation is solved by

$$f(x_1, v_1) = C \exp\left[-\frac{1}{2D}\left(2\alpha_1 v_1^2 + \omega_0^2 x_1^2\right)\right] \tag{53}$$

The dispersion is given by

$$< v_1^2 >\simeq \frac{D}{2\alpha_1}. \tag{54}$$

The longitudinal dispersion depends on the constant α_1 which is still to be determined. For the transversal fluctuations against the center om mass of the swarm the situation is more complicated due to the problems with linear stability in v_2- dirction. Neglecting the coupling to the longitudinal fluctuations which are small we find

$$\frac{d}{dt}v_2 + \omega^2 x_2 = v_2[(\alpha - \alpha_1 - \sigma) - \beta(\delta v_2)^2] + \sqrt{2D_r}\xi_2(t) \tag{55}$$

We remember that $\alpha_1 < \alpha$ for finite noise. Therefore the first term on the r.h.s. may be positive or negative, depending on the situation. We have to differ between two cases:
(i) For $\sigma > (\alpha - \alpha_1)$ we have a standard problem of noisy damped oscillator with the distribution

$$f(x_2, v_2) = C \exp\left[-\frac{1}{2D}\left(-(\sigma - \alpha - \alpha_1)v_2^2 + \omega_0^2 x_1^2\right)\right] \tag{56}$$

and the dispersion

$$< v_2^2 >\simeq \frac{D}{(\sigma - \alpha + \alpha_1)} \simeq \frac{D}{\sigma}. \tag{57}$$

In the present case the fluctuatons of v_1 and v_2 are rather small and the translational mode os most favourable.
A different situation is observed in the in the second case of small (or zero) contribution from parallelizing interactions.
(ii) We consider now the situation $\sigma < (\alpha - \alpha_1)$. We see that now eq.55 corresponds to an equation of a limit cycle. However this limit cycle is not a standard one, since $(\alpha - \alpha_1) > 0$ requires finite noise. We see therefore that the limit cycle is noise driven. The distribtuion of the corresponding distribution for the velocity fluctuations (which are uncoupled in our approximation) is given by

$$f(x_2, v_2) = C \exp\left[-\frac{1}{2D}\left(-(\alpha - \alpha_1 - \sigma^2)H + 2\beta H^2\right)\right] \tag{58}$$

where the effective Hamiltonian of the limit cycle is

$$H(x_2, v_2) = \frac{1}{2}(v_2^2 + \omega_0^2 x_2^2) \tag{59}$$

We remind that this distribution corresponds to a driven motion of the center of mass supplemented by a small oscillatory driven motion against the center of mass. The dispersions are given by

$$< v_1^2 >\simeq \frac{D}{2\alpha_1} \tag{60}$$

$$< v_2^2 >\simeq \frac{\alpha - \alpha_1 - \sigma^2}{2\beta} \tag{61}$$

The dispersions are connected by the relations

$$\alpha_1 = \alpha - \frac{3\beta D}{\alpha_1} + \sigma^2 \tag{62}$$

$$\alpha_1^2 - \alpha_1(\alpha + \sigma^2) + 3\beta D = 0 \tag{63}$$

The quadratic equation has the solution

$$\alpha_1 = \frac{1}{2}(\alpha + \sigma^2)\left[1 - \sqrt{1 - \frac{12\beta D}{(\alpha + \sigma^2)^2}}\right] \tag{64}$$

We see that the dispersion of V^2 is maximal for the critical noise strength

$$D_{cr} = \frac{(\alpha + \sigma^2)^2}{12\beta} \tag{65}$$

For the case $\alpha = \beta = 1, \sigma = 0$ this gives the critical value $D_{cr} \simeq 0.08$. This is in quite good agreement with the value $D_{cr} \simeq 0.07$ found in a simulation for this choice of parameters [12]. We note however that our more recent simulations show that the translational mode is more uunstable than these analytical estimates would suggest. Waiting a sufficiently long time we have seen in many simulations for $\sigma = 0$ that even at such small noise as $D = 0.001$ the system goes finally to the rotational mode (see Figs. 9-10). This uberlines the importance of having at least a small contribution of velocity synchronization $\sigma > 0$. The solutions of the Fokker-Planck equation for the rotational mode are are distributed around two limit cycles corresponding to left or right rotations. These distributions for the rotational mode are similar to what we have found for the case of external fields. The probability is distributed around two limit cycles corresponding to left or right rotations. The time development of the distributions is demonstrated for two sets of parameters Figs. 7 and 8

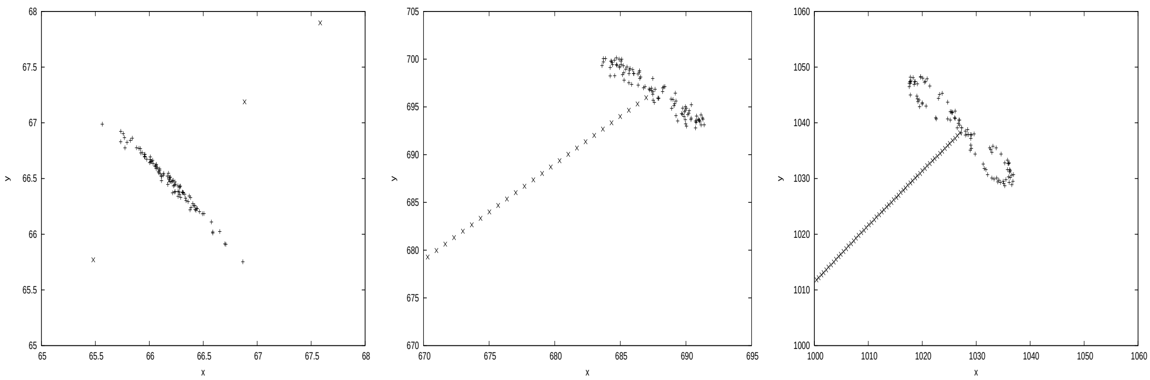

Fig. 7. A time sequence ($t = 100, 100, 1500$) of swarm configurations for very small noise demonstrates the dynamical region of translational motion $V = v_0$ (parameters $\omega_0^2 = 0.01; D = 0.00001; N = 100$).

Summarizing our findings we may state: For two interacting active particles there exist a translational and a rotational mode.
In the rotational mode the center of the "dumb-bell" is at rest and the system is driven to rotate around the center of mass. Only the internal degrees of freedom are excited and we observe driven rotations. In the translational mode of the dumb-bell the center of mass of the

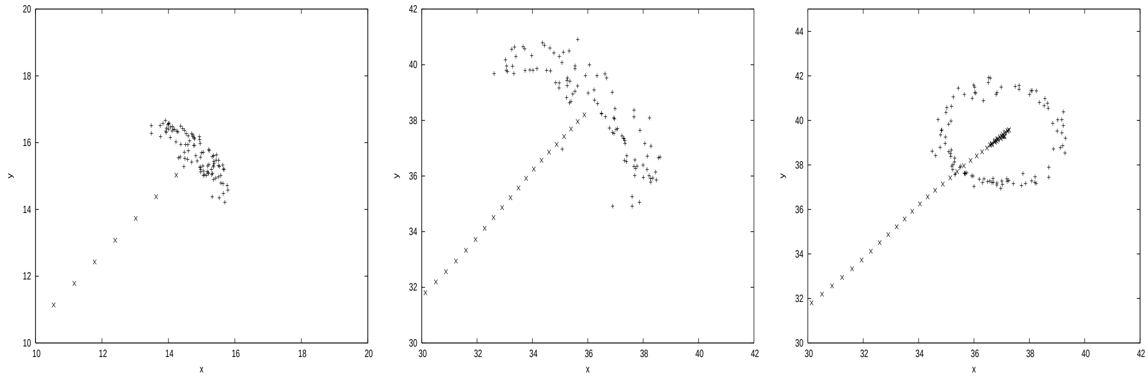

Fig. 8. A time sequence ($t = 20, 80, 110$) of swarm configurations at larger noise demonstrates the dynamical region of rotational motion (parameters $\omega_0^2 = 0.2; D = 0.001; N = 100$).

dumb-bell makes a driven Brownian motion similar to a free motion of the center of mass. In this case we may expect distribution similar as give in Acta Phys. Pol..

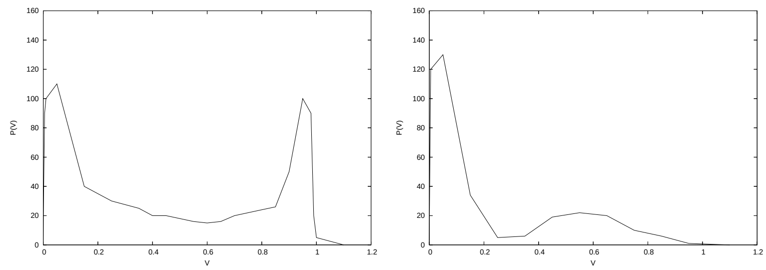

Fig. 9. Bistable distributions of the swarm velocity V estimated from simulations. Left panel: The translational mode produces for small noise $D = 0.00001$ a narrow maximum near to the maximal velocity of the swarm $V = v_0$ (here $v_0 = 1$). Right panel: At larger values of noise $D = 0.01$ dominates the rotational mode which forms a big maximum corresponding to more rotational configurations .

7 Conclusions

We studied here the active Brownian dynamics of swarms of confined particles with velocity-dependent friction, parallelizing dissipative forces and attracting conservative interactions. We have given here first an analysis of several simple cases. This way we could identify several qualitative modes of movement.

First we studied the mode of rigid-oriented body, in which all particles of the swarm move parallel with fixed distances. The center of mass makes translational or - in the presence of an attracting center - rotational motions around a center. This type of motion which holds for the case of strong parallelizing forces. In a next section we study the case of very small (or zero) parallelizing forces and attracting linear spring forces between the particles and no external forces. The case of linear spring forces corresponds to global coupling in coordinate space and the whole problem reduces to the dynamics of pairs. We investigate the attractor structure of the dynamical system for pairs and show the existence of one point attractor corresponding to translation and two limit-cycle attractors corresponding to left/right rotations of the pair. Further we have made a numerical study of N particle systems. In particular we investigated the rotational and translational modes of the swarm. Finally we investigated analytically the noise induced transitions between rotations and translations.

We did not intend here to model any particular problem of biological movement. We note however that the study of dynamic modes of collective movement of swarms may be of some importance for the understanding of many biological and may be even of social collective motions. To support this view we refer again to the books cited at the beginning of this investigation [1–5]. Further we mention the experimental investigations, for expample the studies of the motion of animals in water, for example the collective motion of daphnia [6].

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