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# Convoluted Gauss-Levy distributions and exploding Coulomb clusters*

W. Ebeling[1], M.Yu. Romanovsky[2], I.M. Sokolov[1], and I.A. Valuev[2]

Institut für Physik, Humboldt-Universität zu Berlin
Newtonstraße 15, 12489 Berlin, Germany;
Russian Academy of Science, Branch of Physical Sciences, Moscow, Russia

**Abstract.** We study the kinetics and the distributions of nonequilibrium systems including Gaussian and Levy-type stochastic forces. We develop the assumption that deviations from the Maxwell distribution which are often observed in nonequilibrium systems may be described by convoluted Gauss-Levy distributions. We derive these distributions by solving Langevin and Fokker-Planck equations for the velocities including two noise sources, centrally distributed over Levy and Gauss functions. As an application, we estimate the evolution of the velocity distributions of exploding Coulomb clusters analytically and by simulations. We show the development of a shoulder in the distribution which is typical for convoluted Gauss-Levy distributions.

## 1 Introduction

Levy-type distributions found applications to physical systems since nearly 100 years [1–5]. The stochastic Holtsmark field acting on charged particles in a plasma is just one example of a force with a long tail distribution. It was shown that this is the source of several interesting physical effects [2–4]. Since stochastic fields generate stochastic accelerations of particles moving in the field, we may expect that corresponding velocity and energy distributions with long tails are generated as predicted by several authors [6–15].

In a recent work, we have shown that mixed (convoluted) Gauss-Levy distributions are candidates for describing observed deviations from Maxwell distributions in plasmas and other systems [15–17]. The convolution corresponds to a product in the Fourier space. This way mixing a Gaussian with a Levy component gives a velocity distribution of the type

$$f_{GL}(v;\alpha) = \frac{1}{\pi}\int_0^{+\infty} \cos(vt)\exp\left(-\delta t^{\alpha} - \frac{\tau}{2}t^2\right) dt \tag{1}$$

Here the coefficient $\delta$ gives the strength of the Levy-type contribution. This distribution is rather complicated and depends on 3 parameters. The quantity $\tau$ is a kind of "temperature" of the Gaussian body. Later we will consider some scaling properties. Special cases of this distribution were already considered in the context of distributions for plasmas [15–17]. The corresponding distribution of kinetic energy $\epsilon = mv^2/2$ is given by

$$f(\epsilon;\alpha) = \frac{1}{\pi}\int_0^{\infty} dtt\sin(t\sqrt{\epsilon})\exp\left(-\frac{\tilde{\delta}}{\alpha}t^{\alpha} - \frac{\tilde{\tau}}{2}t^2\right) \tag{2}$$

where $\tilde{\delta} = \sqrt{m/2}\delta$ and $\tilde{\tau} = (m/2)\tau$. In Fig. 1 (left panel) we show the general behavior of the pure and the mixed component of the velocity distribution for the Cauchy case $\alpha = 1$ for different values of the

* Dedicated to the 60th birthday of our friend and colleague Lutz Schimansky-Geier

parameters $\delta$ and $\tau$ as well as the corresponding behavior of the energy distribution for the Cauchy case $\alpha = 1$ using again different values of the parameters $\tilde{\delta}$ and $\tilde{\tau}$. The representation of energy distributions

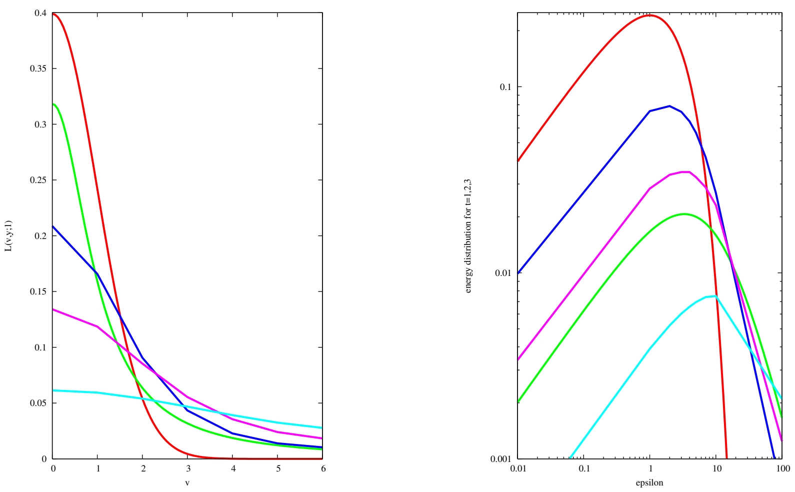


**Fig. 1.** Comparison of a Gauss distribution with Cauchy and convoluted Gauss-Cauchy distributions. Left panel: Gaussian distributions of the velocities ($\delta = 0, \tau = 1$, red line, fastest decay) compared with a Cauchy distribution ($\delta = 1, \tau = 0$, green, slow decay) and 3 convoluted Gauss-Cauchy distributions ($\delta = 1, 2, 5, \tau = 1$, the curves in between in the colors blue, magenta, turquoise). Right panel: Log-log plot of a Gauss distribution of the kinetic energy ($\tilde{\delta} = 0, \tilde{\tau} = 1$, represented by red line, fast decay) compared with a Cauchy distribution ($\tilde{\delta} = 10$, $\tilde{\tau} = 0$, green, slow decay) and 3 convoluted Gauss-Cauchy distributions ($\tilde{\delta} = 1, 2, 5$, $\tilde{\tau} = 1$, the curves in between in colors blue, magenta, turquoise).

in Fig. 1 (right panel) shows that the convoluted Gauss-Cauchy distributions are at least candidates for a description of the experimental data for exploding clusters data which also show a rather long energy tail [18–20]. Further the Figure shows that the theoretical distribution eq. 2 has in the region of transitions between the Gaussian body and the Cauchy tail a typical shoulder.

In some earlier work we already studied the influence of Levy noise on the velocity distribution and discussed applications to rate processes [16]. Further we took into account that in a dense plasma the stochastic forces have a more complicated distribution including a main body of Gaussian character [17]. In the mentioned work we made the simplifying assumption that the real noise in an interacting system, e.g. in a plasma, may be approximated by the sum of a Gaussian noise and a Levy noise. Here the Gaussian noise models the usual small angle scattering of the particles and the Levy noise should model the action of the electrical microfields, turbulent fields etc. and in particular their high field tails. The problem of various noise sources in Langevin equation is well-investigated both for Gauss sources (see, for example the classical work [3]) as well as for Levy sources [6, 7]).
Levy sources of noise are usually connected with super-diffusion and Levy flights. There are many examples of super-diffusion of different origin [10] including Tokamak plasmas [11, 12] and turbulent liquids [13]. The present work was stimulated by these works and by the observation that in many nonequilibrium systems, in particular in turbulent systems [10–13], partially Gaussian distributions with Levy-type wings are found. We mention further the observation of Levy statistics in many-particle quantum systems (hard core bosons) [14]. Our interest is devoted here to systems were the stationary probability distribution functions are of Boltzmann-Gibbs-type with Levy-type wings. One of the possible reasons could be additional noise sources with Levy character. We will consider here systems with two noise sources with Gaussian and Levy-type probability distribution functions (PDF's). As a specific example of Levy distributions we will study Gauss-Cauchy distributions, which are analytically most simple.

Besides the general considerations, we study also some applications to exploding plasmas. This is a particular example of a far from equilibrium system. The energy distributions and other properties were studied experimentally and theoretically by a number of workers [18–21]. The existence of anomalous diffusion in plasmas was observed experimentally at the edge of fusion plasmas [22]. A theoretical explanations was given also by several authors [9, 12] based on the hypothesis of Levy tails due to turbulent effects in the plasma. We do not exclude this effect here but we include also Holtsmark-type stochastic fields as an additional source of long tails. Based on this idea, we studied already in our foregoing work [16, 17] the evolution of the distribution functions and gave several applications to exploding clusters. This line of research will be continued here.

## 2 Velocity and energy distributions for Levy flights

The rate of many elementary processes strongly depends on velocity distributions. Usually one assumes here Maxwell distributions but the appearance of long tails might have a strong influence. The latter will now be discussed within a simple model, taking into account such long tails of stochastic forces as appearing e.g. due to the influence of electric microfields. The amplitude of the local electric field in dilute plasmas is given by a Holtsmark distribution, i.e. by a symmetric Lévy law with exponent $3/2$ [1, 3]. Simulations confirm also the existence of long tails of the distribution of the stochastic electric fields in real dense plasmas [23].

Because of the stochastic character of the fields, we postulate a Langevin equation for the velocities with two statistically independent noise sources [17]. The Langevin equation for the velocity of a particle with the mass $m$ under influence of such fields can be written in the form

$$\frac{dv}{dt} + \gamma_0 v = \frac{F_L(t)}{m} + \frac{F_G(t)}{m} \tag{3}$$

where $F_L(t)$ is the field strength with Levy PDF with short-time autocorrelations, $F_G$ is a force with a Gaussian PDF, and $\gamma_0$ a phenomenological friction coefficient. The two random forces, $F_L(t)$ and $F_G(t)$ are independent random quantities, the first one, $F_L(t)$, being the actual value of the local field (e.g. of an electric microfield), and the second one, $F_G(t)$, characterizing the intensity of collisions. The (stationary) distribution of $F_L(t)$ is

$$W(F_L) = \frac{1}{\pi}\int_0^\infty dk \cos(F_L k)\exp(-k^\alpha \sigma_L) \tag{4}$$

where $0 < \alpha \le 2$, and $\sigma_L$ is the characteristic "field strength" (the scaling parameter of the distribution). For ideal plasmas $\alpha = 3/2$ and $\sigma_L = eE_H = 2\pi e(4/15)^{2/3} q n^{2/3}$, $e$ is the charge of randomly located particles creating the microfield, $n$ is their density [1]. The procedure for the solution of eq. (3) is well-known [3, 6]. The formal solution of eq. (3) is

$$v(t) = v_0 + \exp(-\gamma_0 t)\int_{t_0}^{t} \exp(\gamma_0 t')\left(\frac{F_L(t')}{m} + \frac{F_G(t')}{m}\right) dt' \tag{5}$$

Let us assume first $t_0 = 0$, $\gamma_0 = 0$, as well as $v_0 = 0$ to simplify derivations. By splitting the time interval $(0, t)$ into subintervals $\Delta t_j$ such that

$$\sum_{t_j=0}^{t} (\Delta t_j) = t,$$

one may replace the integral on the right side of (5) by sums of integrals

$$v = \frac{1}{m}\sum_{t_j=0}^{t}\int_{t_j}^{t_j+\Delta t_j} F_L(t')dt' + \frac{1}{m}\sum_{t_i=0}^{t}\int_{t_i}^{t_i+\Delta t_i} F_G(t_i)dt'.$$

Each integral over a $\Delta t_k$-subinterval, $P_L(t_j) = \int_{t_j}^{t_j+\Delta t_j} F_L(t')dt'$ and $P_G(t_i) = \int_{t_i}^{t_i+\Delta t_i} F_G(t_i)dt'$, represents the momentum transfer during the corresponding interval due to the corresponding process. These ones will be considered as independent random variables, possessing the Lévy and the Gaussian distribution, respectively. Here, the presence of two sums means that the time intervals for various noises may be chosen to be different. Note that the Gaussian noise in the standard Langevin equation is usually assumed to be $\delta$-correlated in time, while the Levy noise would normally have some non-zero autocorrelation time, due to the fact that the field which is smooth in the space between charges, and its changes are mostly due to the displacements of charges and of the particle of interest. In what follows the corresponding intervals will be chosen the same and equal to $\Delta t_i = \Delta t_j = \Delta t$. The physically small $\Delta t$ may be chosen to be equal to the corresponding correlation time $\tau_{corr}$. Thus,

$$v = \frac{1}{m}\sum_{j=0}^{N} P_L(t_j) + \frac{1}{m}\sum_{i=0}^{N} P_G(t_i)$$

with $N = t/\Delta t$, i.e. is represented by the sum of independent random variables drawn from two different distributions.

The transferred momenta $P_G$ are distributed according to the Gaussian law,

$$W_G(P_G) = \frac{1}{\sqrt{\pi}P_{G0}} \exp\left(-\frac{P_G^2}{P_{G0}^2}\right) \tag{6}$$

where $P_{G0}$ is the characteristic momentum transferred during the time interval $\Delta t$ due to collisions (we may use the estimate $P_{G0}^2 \simeq (mv)^2 \Delta t \nu$ where $\nu$ is the collision frequency). The momenta transferred due to interaction via fields are distributed according to a Lévy law,

$$W_L(P_L) = \frac{1}{P_{L0}}\mathcal{L}_\alpha\left(\frac{P_L}{P_{L0}}\right)$$

where $\mathcal{L}_\alpha(x)$ is the probability density function of a Lévy distribution of index $\alpha$ given by $L_\alpha(x) = (1/\pi)\int_0^\infty dk \cos(kx)\exp(-k^\alpha)$. We denote the approximate probability distribution function of the velocity $v$, depending on time, by

$$f(v,t) = \left\langle \delta\left(v - \sum_{j=0}^{N} \frac{P_L(t_j)}{m} - \sum_{i=0}^{N} \frac{P_G(t_i)}{m}\right)\right\rangle \tag{7}$$

where $\delta(x)$ is Dirac's delta-function and $N = t/\Delta t$ is the overall number of intervals. The averaging here is done over all random values $P_L(t_j)$ and $P_G(t_i)$. Passing to a Fourier representation of the $\delta$-function, one gets

$$\begin{aligned} f(v,t) &= \frac{1}{2\pi}\left\langle \int_{-\infty}^{\infty} \exp\left(iK\left(v - \sum_{j=0}^{N}\frac{P_L(t_j)}{m} - \sum_{i=0}^{N}\frac{P_G(t_i)}{m}\right)\right) dK \right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iKv)dK \prod_{j=0}^{N}\prod_{i=0}^{N}\left\langle \exp\left(-iK\frac{P_L(t_j)}{m} - iK\frac{P_G(t_i)}{m}\right)\right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iKv)dK \prod_{j=0}^{N}\prod_{i=0}^{N} I_{i,j}(K) \end{aligned} \tag{8}$$

where the integral $I_{i,j}(K)$ equals

$$I_{i,j}(K) = \int_{-\infty}^{\infty} W_L dP_L \int_{-\infty}^{\infty} W_G dP_G \exp\left(-iK\left(\frac{P_L(t_j)}{m} + \frac{P_G(t_i)}{m}\right)\right). \tag{9}$$

Moreover, $I_{i,j}(K) = I_j(K)I_i(K)$, and the integrals $I_j(K)$ and $I_i(K)$ are easily evaluated, and

$$I_{i,j}(K) = \exp\left(-K^{\alpha}\left(\frac{P_{L0}}{m}\right)^{\alpha} - K^2\left(\frac{P_{G0}}{2m}\right)^2\right). \tag{10}$$

This result we had expected since it corresponds to the PDF of a random value that is a sum of two independent random values. In this case, the characteristic function of above random value is the product of characteristic functions of the summed random values. Substituting the expression (10) into (8) one finds

$$\begin{aligned} f(v,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dK \exp\left(iKv - |K|^{\alpha} N \left(\frac{P_{L0}}{m}\right)^{\alpha} - K^2 N \left(\frac{P_{G0}}{2m}\right)^2\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dK \exp\left(iKv - |K|^{\alpha} t \left(\frac{\Delta t^{\frac{\alpha-1}{\alpha}} F_{L0}}{m}\right)^{\alpha} - K^2 t \left(\frac{F_{G0}\Delta t^{1/2}}{2m}\right)^2\right) \end{aligned} \tag{11}$$

where $F_{G0} = P_{G0}/\Delta t$ and $F_{L0} = P_{L0}/\Delta t$ are the typical forces of the corresponding nature acting during the $\Delta t$ intervals. This result coincides with the result for free Levy flights for $F_{G0} = 0$ [6]. For $\alpha = 1$ and $F_{G0} = 0$ our procedure leads to the well-known (time-dependent) Cauchy distribution introduced above:

$$f(v,t;\alpha = 1, F_{G0} = 0) = \frac{(F_{L0}/m)t}{\pi(v^2 + (F_{L0}t/m)^2)} \tag{12}$$

For $\gamma_0 = 0$, the PDF $W(v,t)$ is broadening with time. This means that our PDFs may physically describe only nonstationary situations. In order to bring the distribution to a stationary state we have to introduce dissipation. We will use as a first approximation a Langevin equation with a friction term $\gamma_0 > 0$. Now we repeat the procedure with some changes. We split the time interval $0, t$ into small time intervals $\Delta t_j$ of length $\Delta t$. Then the velocity $v$ can be written as

$$v \simeq \sum_{j=0}^{t/\Delta t} \exp(-\gamma_0(t - t_j))\frac{P_L(t_j)}{m} + \sum_{i=0}^{t/\Delta t} \exp(-\gamma_0(t - t_i))\frac{P_G(t_i)}{m} \tag{13}$$

and differs from the previous case only in the fact that the contributions of the corresponding transferred momenta to the final velocity decrease with time. Repeating the same procedures which were used in (7) and (8) to determine the $W_f(v,t)$, one finds the PDF for the Langevin equation with dissipation :

$$\begin{aligned} f(v,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iKv) dK \prod_{j=0}^{t/\Delta t}\prod_{i=0}^{t/\Delta t} I_{i,j}(K) \\ &= \exp\left(-K^{\alpha}\left(\frac{P_{L0}}{m}\right)^{\alpha}\sum_{j=1}^{N}\exp(-\alpha\gamma_0(t-t_j)) - K^2\left(\frac{P_{G0}}{2m}\right)^2\sum_{i=1}^{N}\exp(-2\gamma_0(t-t_i))\right) \\ &\simeq \exp\left(-K^{\alpha}\left(\frac{P_{L0}}{m}\right)^{\alpha}\frac{1-\exp(-\alpha\gamma_0 t)}{\alpha\gamma_0} - K^2\left(\frac{P_{G0}}{2m}\right)^2\frac{1-\exp(-2\gamma_0 t)}{2\gamma_0}\right), \end{aligned} \tag{14}$$

the only difference to the previous case being that the integrals $I_{i,j}(K)$ depend on $K$ via $t_i$ and $t_j$. For small $t$ the result reduces to eq.(10). For large $t$, the characteristic function becomes stationary and produces a stationary PDF $f(v, t \to \infty)$. For pure Gaussian noise we expect a Gaussian PDF for the velocity :

$$f(v) = \frac{1}{\sqrt{\pi}v_{T0}}\exp\left(-\frac{v^2}{v_{T0}^2}\right)$$

where $v_{T0}$ is a corresponding equilibrium "thermal" velocity. It means that

$$\frac{P_{G0}}{m\sqrt{2\gamma_0}} = v_{T0} \tag{15}$$

By including the Levy term the procedure provides the final PDF in form of a Fourier transform:

$$\begin{aligned} f(v,t) &= \frac{1}{\pi}\int_0^\infty dK\cos(Kv)\cdot \\ &= \exp\left(-K^\alpha\delta_0\frac{1-\exp(-\alpha\gamma_0 t)}{\alpha\gamma_0} - \beta_0 K^2\frac{1-\exp(-2\gamma_0 t)}{2\gamma_0}\right) \end{aligned} \tag{16}$$

with

$$\delta_0 = \Gamma(\alpha)(P_{L0}/m)^\alpha \tag{17}$$

and

$$\beta_0 = (\gamma_0 v_{T0})^2/2. \tag{18}$$

Our result may be written in the form of eq. (1) with the only difference that the parameters are now time dependent

$$f(v;\beta,\delta,\alpha) = \frac{1}{\pi}\int_0^{+\infty}\cos(vK)\exp[-\delta(t)K^\alpha - \beta(t)K^2]dK \tag{19}$$

Here the coefficient $\delta(t)$ gives the time-dependent strength of the Levy-type contribution. The distribution which we obtained this way is rather complicated and depends on 3 parameters $\delta(t), \beta(t), \alpha$ defined by

$$\delta(t) = \delta_0\frac{1-\exp(-\alpha\gamma_0 t)}{\gamma_0\alpha}, \tag{20}$$

$$\beta(t) = \beta_0\frac{1-\exp(-2\gamma_0 t)}{2\gamma_0}. \tag{21}$$

By introducing a kind of "temperature" $\tau(t) = 2\beta(t)$ corresponding to the "Gaussian body" of the distribution, we may use the scaling

$$f(v;\beta,\delta,\alpha) = \frac{1}{\sqrt{\tau(t)}}F\left(\frac{v}{\sqrt{\tau(t)}};\frac{\delta(t)}{\sqrt{\tau(t)}},\alpha\right) \tag{22}$$

with the new function of a dimensionless velocity $v/\sqrt{\tau(t)}$ and a parameter $y(t) = \delta/\sqrt{\tau(t)}$ which is defined by

$$F(x;y(t),\alpha) = \frac{1}{\pi}\int_0^{+\infty} dz\cos(xz)\exp[-y(t)z^\alpha - t^2/2] \tag{23}$$

As time $t$ increases, the PDF (23) changes from the form (11) of the fully non-stationary PDF to the stationary PDF where $t\to\infty$ which agrees with eq. (1) of section 1, obtained by convolution.

## 3 Generalized Chandrasekhar equation for Levy and Gaussian noise

For solving concrete problems it is useful to formulate kinetic equations following the lines of Klein, Kramers, and Chandresekhar. Studying first the spatially uniform 3D-case, we generalize the approach developed by Fogedby et al. for the case of pure Levy noise [6,7,9]. This way we get the fractional Fokker-Planck equation corresponding to our Langevin equation

$$\frac{\partial}{\partial t}f(\mathbf{v},t) = \frac{\partial}{\partial v}\left(\gamma_0\mathbf{v}f(\mathbf{v},t)\right) + D_\alpha\nabla^\alpha f(\mathbf{v},t) + D_2\nabla^2 f(\mathbf{v},t) \tag{24}$$

with $D_2 = k_B T\gamma_0/m$, a space-fractional distributed-order diffusion equation, see Ref.[24]. In Fourier space we find

$$\frac{\partial}{\partial t} f(\mathbf{s},t) = -\gamma_0 \mathbf{s}\cdot\frac{\partial}{\partial \mathbf{s}} f(\mathbf{s},t) - D_\alpha |\mathbf{s}|^\alpha f(\mathbf{s},t) - D_2\mathbf{s}^2 f(\mathbf{s},t) \tag{25}$$

For short times we may neglect the term containing the friction and obtain the solution

$$f(\mathbf{s},t) = f(\mathbf{s},0)\exp[-D_\alpha|\mathbf{s}|^\alpha t - D_2\mathbf{s}^2 t] \tag{26}$$

We note that already the short-time distribution shows Levy tails, examples for different $\alpha$ and different times are shown in Fig. 2.
Further the full time-dependent solution in $s$-space is known which reads

$$f(\mathbf{s},t) = f(\mathbf{s},0)\exp\left[-\frac{D_\alpha|\mathbf{s}|^\alpha}{\alpha\gamma_0}\left(1-\exp(-\alpha\gamma_0 t)\right) - \frac{D_2\mathbf{s}^2}{2\gamma_0}\left(1-\exp(-2\gamma_0 t)\right)\right] \tag{27}$$

In the asymptotic case the solution the stationary solution reads

$$f_0(\mathbf{s},\infty) = C\exp\left[-\frac{D_\alpha}{\alpha\gamma_0}|\mathbf{s}|^\alpha - \frac{D_2}{2\gamma_0}\mathbf{s}^2\right] \tag{28}$$

where the constant $C$ is given by the normalization. This distribution has a diverging mean square, i.e. the velocity distribution has a long tail. For the distribution of the modulus of the velocity we get

$$F_0(|v|,t) = \frac{2|v|}{\pi}\int_0^\infty y\sin(|v|y)\exp(-\delta(t)y^\alpha - \frac{\tau(t)}{2}y^2)dy \tag{29}$$

The asymptotics is in the general case given by

$$F_0(\mathbf{v}) \sim \frac{k_B T}{m\alpha|\mathbf{v}|^{\alpha+1}}. \tag{30}$$

Applications of these formulae to cell motility seem to be possible. We mention for example several observations supporting the hypothesis that the search strategy of cells makes use of mechanisms of Levy flight and generates Levy-type velocity distributions. For example, Li et al. [25] found non-Gaussian long tails in the motion of eukariotic cells. Lewandowsky et al. [26] proposed Levy walks as a model for ameboid motion and showed that Levy walks provide a more efficient search strategy than Brownian motion, and in recent work Bödeker et al. [27] analyzed data for ameboid motion finding clear evidence for non-Gaussian heavy tails.

## 4 Generalized Chandrasekhar equations for three-dimensional domains

Let us generalize now the stochastic equations for the case that the particle is subject to Gaussian and Levy noise including also space-dependence and external fields, i.e. we take all classical Chandrasekhar terms into account. We will study now domains, meaning mesoscopic spatial subsystems with quickly changing microscopic velocities but slowly changing fields and spatial distributions. The idea is, that deviations from the Maxwell distribution may be of special relevance for such mesoscopic domains with slowly changing fields [15]. For the distributions within a domain, we obtain the fractional stochastic equation

$$\frac{\partial}{\partial t} f(\mathbf{r},\mathbf{v},t) + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} f(\mathbf{r},\mathbf{v},t) + \frac{\mathbf{F}}{m}\cdot\frac{\partial}{\partial \mathbf{v}} f(\mathbf{r},\mathbf{v},t) \tag{31}$$

$$= \frac{\partial}{\partial v}\left(\gamma_0 \mathbf{v} f(\mathbf{r},\mathbf{v},t)\right) + D_\alpha\nabla^\alpha f(\mathbf{r},\mathbf{v},t) + D_2\nabla^2 f(\mathbf{r},\mathbf{v},t) \tag{32}$$

with $D_2 = k_B T\gamma_0/m$. After transformation to Fourier space we find

$$f(\mathbf{k},\mathbf{s},t) = \int\frac{d\mathbf{r}d\mathbf{v}}{(2\pi)^6} exp(i\mathbf{k}\cdot\mathbf{r} + i\mathbf{s}\cdot\mathbf{v}) f(\mathbf{r},\mathbf{v},t) \tag{33}$$

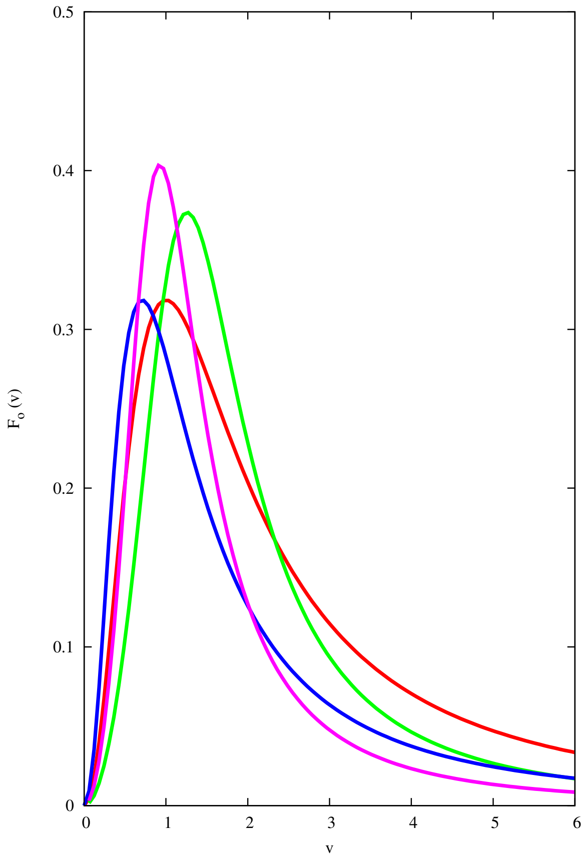


**Fig. 2.** Evolution of the probability distribution of the modulus of the velocity $F_0(|v|)$ for 2 different Levy indices $\alpha = 1$ and $\alpha = 3/2$ at $t = t_0$ (red and green curves) and the same at a later time $2t_0$ (blue and violet curves).

and

$$\frac{\partial}{\partial t} f(\mathbf{k}, \mathbf{s}, t) - \mathbf{s} \cdot \frac{\partial}{\partial \mathbf{k}} f(\mathbf{k}, \mathbf{s}, t) - \frac{1}{m} \mathbf{F} \cdot \mathbf{s} f(\mathbf{k}, \mathbf{s}, t) \tag{34}$$

$$= -\gamma_0 \mathbf{s} \cdot \frac{\partial}{\partial \mathbf{s}} f(\mathbf{k}, \mathbf{s}, t) - D_\alpha |\mathbf{s}|^\alpha f(\mathbf{k}, \mathbf{s}, t) - D_2 \mathbf{s}^2 f(\mathbf{k}, \mathbf{s}, t) \tag{35}$$

This equation cannot be valid for all times and spatial domains since the Maxwell distribution

$$f_{Maxwell} = f_0(\mathbf{s}) = const \exp[-\frac{1}{4} v_T^2 \mathbf{s}^2] = const \exp[-\frac{k_B T s^2}{2m}] \tag{36}$$

appears to be a solution only in the special case on a uniform system with $F = 0, D_\alpha = 0$ and with the validity of an Einstein relation

$$D_2 = \frac{\gamma_0}{2 v_T^2} = \frac{k_B T \gamma_0}{m} \tag{37}$$

For the given reasons we assume the validity of the generalized Fokker-Planck equation defined by eq.(35) just for specific domains of size $r_D$ (Debye radius) and $k \simeq k_D = 2\pi / r_D$. Then we have for a domain the approximate relation

$$\frac{\partial}{\partial t} f(\mathbf{k_D}, \mathbf{s}, t) - \frac{e}{m} \mathbf{E} \cdot \mathbf{s} f(\mathbf{k_D}, \mathbf{s}, t) \tag{38}$$

$$= -\gamma_0 \mathbf{s} \cdot \frac{\partial}{\partial \mathbf{s}} f(\mathbf{k_D}, \mathbf{s}, t) - D_\alpha(k_D) |\mathbf{s}|^\alpha f(\mathbf{k_D}, \mathbf{s}, t) - D_2 \mathbf{s}^2 f(\mathbf{k_D}, \mathbf{s}, t) \tag{39}$$

The solution of this equation reads

$$f(\mathbf{k_D}, \mathbf{s}, t) = f(\mathbf{k_D}, \mathbf{s}, t_0) \exp[-id_1(t-t_0)(e/m)\mathbf{E}\cdot\mathbf{s} - d_\alpha(t-t_0)D_\alpha(k_D)|\mathbf{s}|^\alpha - d_2(t-t_0)D_2\mathbf{s}^2] \quad (40)$$

where

$$d_\alpha(t) = \frac{1}{\alpha\gamma_0}[1 - \exp(-t\alpha\gamma_0)], \qquad d_2(t) = \frac{1}{2\gamma_0}[1 - \exp(-2t\gamma_0)] \quad (41)$$

We note that $E$ is the constant part of the electric field in the domain. In the case of weak fields, we may write the Fourier back transform as

$$f(\mathbf{k_D}, \mathbf{v}, t) = \left[1 - \frac{e}{m}E \cdot \frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{k_D}, \mathbf{v}, t, E = 0) \quad (42)$$

For simplicity, we study from now on only situations with $\mathbf{E} = \mathbf{0}$ and $f(\mathbf{k_D}, \mathbf{v}, t_0) = \delta(\mathbf{v} - \mathbf{v(0)})$. Then under the assumption of a Cauchy-Lorentz index $\alpha = 1$ and $D_2 = 0$ the zero field distribution reads

$$f(\mathbf{k_D}, \mathbf{v}, t, E = 0) = \frac{v_0}{\pi^2((\mathbf{v} - \mathbf{v(0)})^2 + v_0(t)^2)} \quad (43)$$

where

$$v_0(t) = d_1(t)\frac{F_{L0}}{m}, \qquad d_1(t) = \frac{1}{\gamma_0}\left[1 - \exp(-\gamma_0(t - t_0))\right]. \quad (44)$$

For Coulomb systems, we get $F_{L0} = eE_H = 2.6e^2n^{2/3}$ and in the case of mixed Cauchy-Lorentz distributions, we find generalized error functions. The Cauchy-Lorentz distribution function of the kinetic energy $\epsilon = mv^2/2$ reads

$$F(\epsilon, \mathbf{k_D}, t) = \frac{2}{\pi}\frac{\sqrt{\epsilon\epsilon_0(t)}}{[\epsilon + \epsilon_0(t)]^2} \quad (45)$$

where

$$\epsilon_0(t) = \frac{F_{L0}^2}{2m}d_1(t)^2 \quad (46)$$

This means that the asymptotics is determined by the Levy contribution.

## 5 Applications to the velocity and energy distribution of ions in exploding clusters.

In the present section, we will give an application to exploding Coulomb clusters following the ideas presented in [15–17]. In some sense the conditions inside a cluster correspond to the domains we studied in the previous section.

This way, we aim to contribute to the explanation of fusion processes in Coulomb clusters, where a considerable neutron yield was observed [18, 19, 28–33]. To our knowledge these experimentally well studied phenomena are only partially understood. We study here the velocity distributions in clusters, in particular, the possibility to enhance nuclear fusion by non-equilibrium distributions of the velocities.

Clusters of charged particles (Coulomb clusters) may be created by powerful laser pulses acting on clusters molecules of hydrogen, deuterium etc. The short range Coulomb repulsion between the nuclei give rises to strong accelerations and the cluster explodes in a time interval between femto- and picoseconds [19, 20]. In order to start with an elementary picture, let us look at a molecule of $H_2$ or $D_2$. Imagine that the electrons are stripped off by a strong laser field. Then instead of the molecule we have two Coulomb particles with distance around $(1 - 2)a_B$ (Bohr radius). The repulsion energy is around 20 eV, for 2 He nuclei, it is around $100eV$, for heavier nuclei, it may be around one keV. If this energy could be converted into stochastic velocities, potentially, it may provide fusion. Let us assume that we have a spherical cluster consisting of some hydrogen-containing material as droplets of liquid hydrogen or deuterium or droplets of water or some hydrocarbons. The part of electrons were removed by a strong laser pulse. The cluster consists of nuclei of rather high density. The initial distances of the nuclei $r_0$ are in part rather small, around a Bohr radius $r_0 \simeq a_B$ corresponding to the original atomic or molecular configurations, which are still present at our initial time $t = 0$. The initial potential energy $Z_1 Z_2 e^2/r_0$ will be converted into kinetic energy of stochastic motion

$$\frac{v_m^2}{2} = \frac{Z_1 Z_2 e^2}{m r_0}. \tag{47}$$

The directions of the velocities are stochastic due to seldom collisions. Then the particle energies may reach at best $100eV$. This estimate shows that higher energies may be reached only by stochastic subsequent accelerations through correlated non-central collisions. We will study here mechanisms based on long tails of the energy and velocity distributions and present some molecular dynamic simulations, which seemingly support our assumptions. The study of plasma clusters and, in particular, fusion in plasma clusters is a rather hot topic [18–21,29,32,33]. We correct and extend now our earlier investigations [15–17]. We have to distinguish between small clusters of size $N \sim 10000$ and large clusters with $N \sim 10^6$ [19,29]. We have shown theoretically and by simulation that even in finite clusters the long range distribution of Holtsmark-type is still well expressed [16]. Before going to simulations, let us start with some analytical estimates for the expansion. Theoretical studies of the field fluctuations in finite systems were given in [15,16]. Here we need the distribution function of an electric microfield in clusters. We will consider only a definite intermediate stage. We assume that nearly all electrons have been ionized and distributed over the plasma. The remaining electron density inside the cluster is denoted by $n_e$ and the ion density by $n$ with $n \geq n_e$. We are interested mainly in the beginning of ion cluster "life", when the cluster density is still very high but already decreasing in time. In this stage, the cluster is hold together only by effects of inertia. Later, the Coulomb explosion is observed, the effective radius

$$R(t) = \int_0^t d\tau V(\tau) \tag{48}$$

increases in time, and the mean density of nuclei in the cluster

$$n(t) = \frac{3N}{4\pi R(t)^3} \tag{49}$$

decreases in time.
After some initial stage the radius increases nearly linear in time [19,20]
As an estimate of zeroth order we may describe the expansion by the law

$$R(t) = R(0)\left[\frac{t}{t_{exp}} + \exp\left(-\frac{t}{t_{exp}}\right)\right] \tag{50}$$

where $t_{exp}$ is a typical time for the first stage of the expansion process which is proportional to the ion plasma frequency

$$t_{exp} = \sqrt{\frac{3}{2}}\sqrt{\frac{m}{4\pi n e^2}}. \tag{51}$$

In the stage of ion clusters, the positions of ions change stochastically in time due to the random fields. Therefore, clusters of ions will be modeled here as systems of randomly distributed charged particles developing under random forces and increasing radius. Let us consider a system of $N$ identical charged particles, randomly located in a spherical volume $V$, which is the model of our physical clusters.
In addition to estimates used in earlier work, we used here the tool of molecular dynamical simulations. We started with a configuration of $N \simeq 100$ protons and $N \simeq 100$ electrons in a dense configuration, similar to a cluster of hydrogen molecules. The interactions are modeled by Coulomb and effective forces, no binding (dispersion forces) were included. The starting velocity distribution corresponds to a temperature around $T = 100$ eV.
Looking at Fig. 3, we see that the expansion of the radius is well described by eq. (50) with $t_{exp} = 2fs$. An interesting point is that the kinetic energy and the potential energy show in the first femtoseconds a maximum and several strongly damped oscillations. Studying the dynamics of the electrons and nuclei

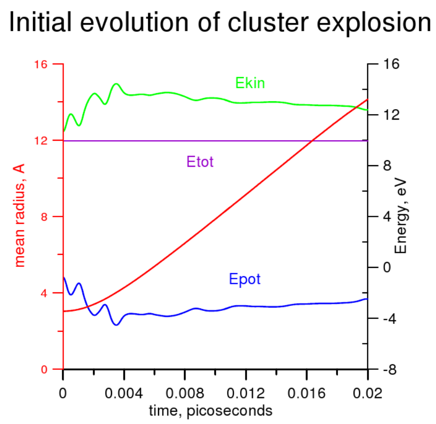


**Fig. 3.** Evolution of the radius, the total energy, as well as the mean kinetic and mean potential energy over time within 20 fs according to molecular dynamic simulations.

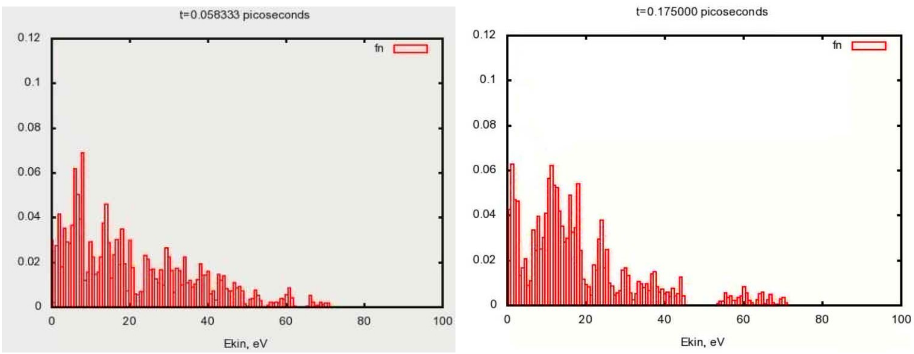


**Fig. 4.** Evolution of the energy distribution of an exploding cluster of 100 hydrogen atoms. We show the distribution at $58fs$ and $175fs$ after the start.

in this stage, we see that most of the expanding electrons are bouncing several times in radial direction back and forth within the the cluster of nuclei before a quasi-steady regime of expansion is reached. Evidently, in this oscillating regime of the electrons, the kinetic energy is strongly increased at the cost of the potential energy and the long tails of the distribution are created. We believe that this might be connected with the celebrated Fermi acceleration effect, an effect which according to Fermi's idea might be responsable for the creation of high energetic particles (cosmic rays) in the Universe [34]. This point, however, is just a speculation and needs further investigation.

Here we are mainly interested in the energy distribution. The snapshots presented in Fig. 5 show strongly fluctuating in time energy tails reaching up to about $70eV$. The appearance is quite similar as found by simulations by Heydenreich et al. [21] for large Xe-clusters. By averaging over many runs for the same time studying a larger number of $N = 200$ protons and electrons, we found the more smooth distributions presented in Fig. 5. In Fig. 6, we used a logarithmic plot of several theoretical curves in comparison with our simulations and experimental values for Xe-clusters obtained in [18], which were discussed in detail in [19,20]. We see that the "best fitted" Boltzmann distribution (where $\tau = 2$) is inappropriate since the long tail of the data is missing. The mixed Gauss-Cauchy distributions fit the data much better, e.g. Gauss-Levy with $\tau = 0.5, \delta = 1$ is near to the experimental data for Xe-clusters the Gauss-Levy distribution with $\tau = 0.5$ and $\delta = 1$. Note that a best fit by adapting $\tau, \delta, \alpha$ makes no sense

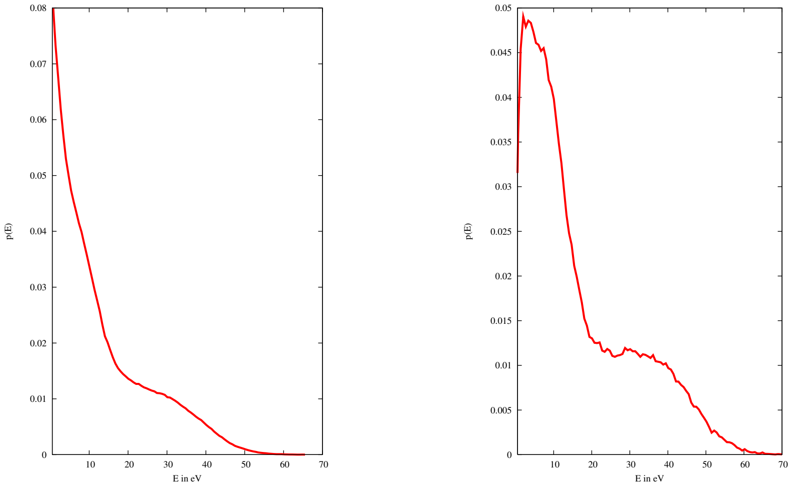


**Fig. 5.** Energy distribution averaged over many runs for a cluster with $N = 200$ protons in an the exploding cluster at two subsequent times in the early stage of the evolution ($t = 20fs$ and $t = 40fs$). We see the development of a ”shoulder” as typical for convoluted Gauss-Levy distributions.

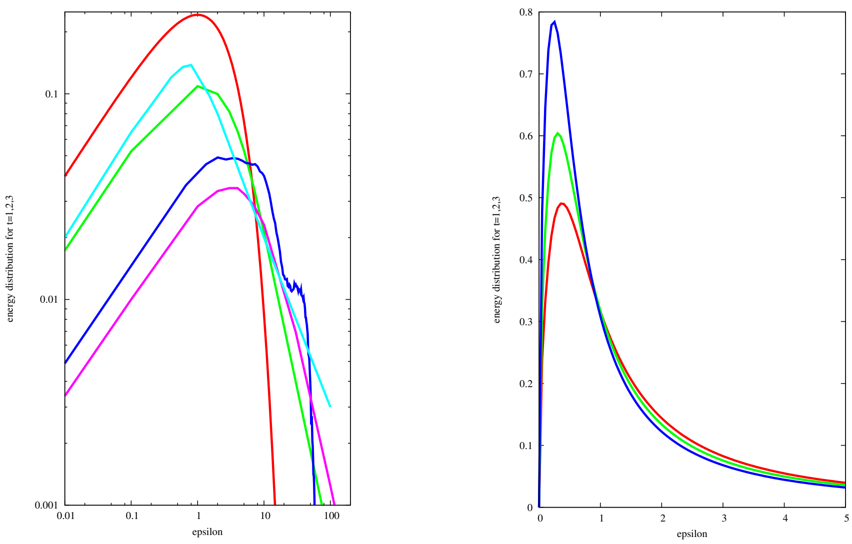


**Fig. 6.** Energy distributions for Coulomb clusters. Left panel: Log-log-plot of several distributions. Besides our simulation result for a cluster with $N = 200$ protons in an exploding cluster at $t = 40fs$ (the curve with a fluctuating shoulder) we present the experimental data of Ditmire et al. for Xe-clusters (turquoise, second from above curve), a Maxwell-Boltzmann distribution ($\delta = 0, \tau = 2$, upper curve) and two Gauss-Levy distributions ($\delta = 1, \tau = 0.5$, green; $\delta = 2, \tau = 1$, magenta). Right panel: Theoretical estimate of the time evolution of the energy distribution at $t = 1t_{exp}$ (red curve) $t = 2t_{exp}$ (green curve) and $t = 3t_{exp}$ (blue curve) assuming $1/\gamma_0 = 2t_{exp}$. The fat tail decreases with increasing time and approaches a Gauss-like tail.

here since the data are not very accurate and our theory gives at best an estimate. Therefore we looked only for a qualitative agreement. In order to compare the time-dependence observed in the simulations with the theory developed, we assumed for simplicity that the distributions may be described by Cauchy-Lorentz distributions with $\alpha = 1, \delta = 1, \tau = 0$. Then the time-dependent energy distributions read [16]

$$F(\epsilon, 1, t) = \frac{2}{\pi} \frac{\sqrt{\epsilon\epsilon_0} g(t)}{\left[\epsilon + \epsilon_0 g(t)^2\right]^2} \tag{52}$$

where the characteristic energy is of the order

$$\epsilon_0 = \frac{e^2 E_H^2}{2m} t_{exp}, \tag{53}$$

$$g(t) = \frac{[1 - \exp(-\gamma_0 t)]}{\gamma_0 [t + t_{exp} \exp[t/t_{exp}]]} \tag{54}$$

Here we the function $g(t)$ takes into account the transition to the quasistationary state and the fact that the density decreases in time due to the expansion. We demonstrated in the right panel of Fig. 6 that the fat tail decreases with increasing time and approaches more and more a Gauss-like parabolic tail. Qualitatively, the simulation results show a similar behavior, a quantitative comparison, however, is not possible at present time. In conclusion, we may say that in exploding clusters as a consequence of correlated collisions, sometimes very energy-rich particles may appear. Possibly these energy-rich particles enhance the fusion rates and are at least one factor which contributes to the observed high neutron rates.

## 6 Conclusions

We have shown that convoluted Gauss-Levy distributions and in particular Gauss-Cauchy distributions may be appropriate candidates for the description of non-equilibrium processes with long tail energy distributions. In particular, we considered as an application the energy distributions of exploding Coulomb clusters in comparison with molecular dynamical simulations. A quantitative comparison is impossible at present time because of the complexity of the simulations. We find, however, several hints pointing to the existence of transient long tails in the energy distribution of exploding Coulomb clusters, which correspond to the transient appearance of very energy-rich particles.

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## References

1. J. Holtsmark, Ann.Physik **58**, 577 (1919); Physik. Z. **20**, 162 (1919), **25**, 73 (1924)
2. D.R. Inglis and E. Teller, Astrophys. J. **90**, 439 (1939).
3. S. Chandrasekhar, Rev. Mod. Phys **15**, 3 – 100 (1943)
4. S. Chandrasekhar and J. von Neumann, Astrophys. J. **95**, 489 (1943)
5. M. Yu. Romanovsky, W. Ebeling, and L. Schimansky-Geier, *J. Phys.: Conf. Ser.*, **11**, 99 (2005).
6. H.C. Fogedby, *Phys. Rev. Lett.* **73**, 2517 (1994); arXiv cond mat 942042v1 Feb 94
7. S. Jespersen, R. Metzler, and H.C. Fogedby, Phys. Rev. E **59**, 2736 (1999); ArXiv 9810176 v1, [cond-mat. stat.-mech.], 15 October 1998.
8. M.Yu.Romanovsky and W.Ebeling, Physica A **252**, 488 (1998).
9. A.V.Chechkin, V.Yu.Gonchar, R.Gorenflo, N.Korabel, and I.M.Sokolov, Phys. Rev. E **78**, 021111 (2008); A.V.Chechkin, R.Gorenflo, and I.Sokolov, Phys. Rev. E **66**, 046129 (2002).
10. R. Metzle and J. Klafter, Phys. Rep. **195**, 127 (2000)
11. A.V. Chechkin and V.Yu. Gonchar, Chaos, Solitons & Fractals **11**, 2379 (2000)
12. A.V. Chechkin and V.Yu. Gonchar, M. Szydlowski, Phys. Plasmas **9**, 78 (2002)
13. K. Tamura, Y. Hidfaka, Y. Yusuf, and S. Kai, Physica A **306**, 157 (2002)
14. A. V. Ponomarev, S. Denisov, and P. Hänggi, arXiv:0907.4328v2, 27 July 2009.
15. M.Yu. Romanovsky and W. Ebeling, Contr. Plasma Phys. **46**, 195 (2006)
16. W. Ebeling and M. Yu. Romanovsky, *Contrib. Plasm. Phys.* **49**, 479 (2009),
17. W. Ebeling, M. Yu. Romanovsky, and I.M. Sokolov *Contrib. Plasm. Phys.* **49**, 704 (2009)
18. T. Ditmire et al., Phys. Rev. A 57, 369 (1998)
19. V.P. Krainov and B.M. Smirnov, Uspheki Fis. Nauk **170**, 969 (2000); Phys.Rep. **370**, 237 (2002).

20. V.P. Krainov, B.M. Smirnov, and M.B. Smirnov, Physics - Uspheki **50**, 907 (2007).
21. A. Heidenreich, I. Last, and J. Jortner, Phys. Chem. Chem Phys. **11**, 111 (2009).
22. B. A. Carreras, D. Newman, and V.E. Lynch, P.H. Diamond, Plasma Phys. Rep. 22, 740 (1996); Phys. Plasmas 3, 2903 (1996); B.A. Carreras et al. Phys. Plasmas **5**3632 (1998), Phys. Rev. Lett. 80, 4438 (1998), 83, 3653 (1999).
23. S. Sadykova, W. Ebeling, I. Valuev, and I. Sokolov, Contrib. Plasm. Phys. **49**, 76 (2009); **50**, 76 (2010).
24. I.M. Sokolov, A.N. Chechkin and J. Klafter, Acta Phys. Pol. B **35** 1323 (2004)
25. L. Li, S.F. Norrelykke, and E.C. Cox, PLoS ONE **3**, e2093 (2008)
26. M. Lewandowsky, B.S. White, and F.L. Schuster, Acta Protozool. **36**, 237 (1997)
27. H.U. Bödeker, C. Beta, T.D. Frank, and E. Bodenschatz, Eur. Phys. Lett. **90**, 28005 (2010)
28. O.F.Hagena, Surf. Science **106**, 101 (1982); O.F. Hagena, Rev. Sci. Instrum. **63**, 2374 (1992).
29. A.Yu. Faenov *et.al.*, Zh. Eksp. Teor. Fiz. **115**, 2051 (1999).
30. T. Ditmire, J. Zweiback, and V.P. Yanovsky *et al*, Phys. Plasmas. **7**, 1993 (2000).
31. J. Zweiback, T.E. Cowan, and T. Ditmire *et al*, Phys. Rev. Lett. **85**, 3640 (2000).
32. D. Hilscher, O. Berndt, and M. Enke *et al*, Phys. Rev. E **64**, 1234 (2001).
33. U. Vogt, H. Stiel, I. Will, P. Nickles *et al*, Appl. Phys. Lett. **79**, 2336 (2001).
34. E. Fermi, Phys. Rev. **75**, 1169 (1949); A.K. Karlis et al. Phys. Rev. E **76**, 016214 (2007)